Jacobi White Paper

Arithmetic vs Geometric: What's it all mean?

Introduction

- Many traditional portfolio modelling approaches were designed to solve a one period portfolio optimization problem. However, these approaches are unsuited to retirement funding problems where sequencing risk, variable cash flows, different tax treatments in accumulation vs decumulation, and other time-dependent factors influence outcomes.
- Moving to a multi-period modelling framework helps solve these problems, but can produce some unexpected results. For instance, it is commonly assumed that the multi-period, geometric return for a multi-asset portfolio can be calculated in the same way as the one period result, by multiplying the average geometric return of the assets by their portfolio weights. This is not necessarily the case.
- While the one period expected return is an intuitive expectation for a multi-period geometric portfolio return, it is not an investable outcome. In a simulation-based analysis both rebalanced and non-rebalanced portfolios are expected to produce a higher return.
- This note explores why this outcome arises, and how some of the key assumptions used in the asset class modelling impact the magnitude of this effect.

1. Expected returns in multi-period, single-asset models

Volatility drag (or variance drain) refers to the effect whereby an asset/portfolio's multi-period expected geometric return is lower when the volatility of the underlying asset/portfolio is higher. A commonly used approximation for this effect is shown in the equation below,

$$E(r_{geo}) \approx E(r) - \frac{1}{2}\sigma^2$$

where r is the arithmetic (single period) expected return, and σ is the volatility of the asset.

The formula shows the reduction to expected geometric return as the asset volatility increases. Notably, this approximation tends to underestimate the actual geometric mean in practical applications. However, the quality of the approximation improves as the number of periods in the average increases, and when estimating the geometric mean for lower expected return, lower variability assets.

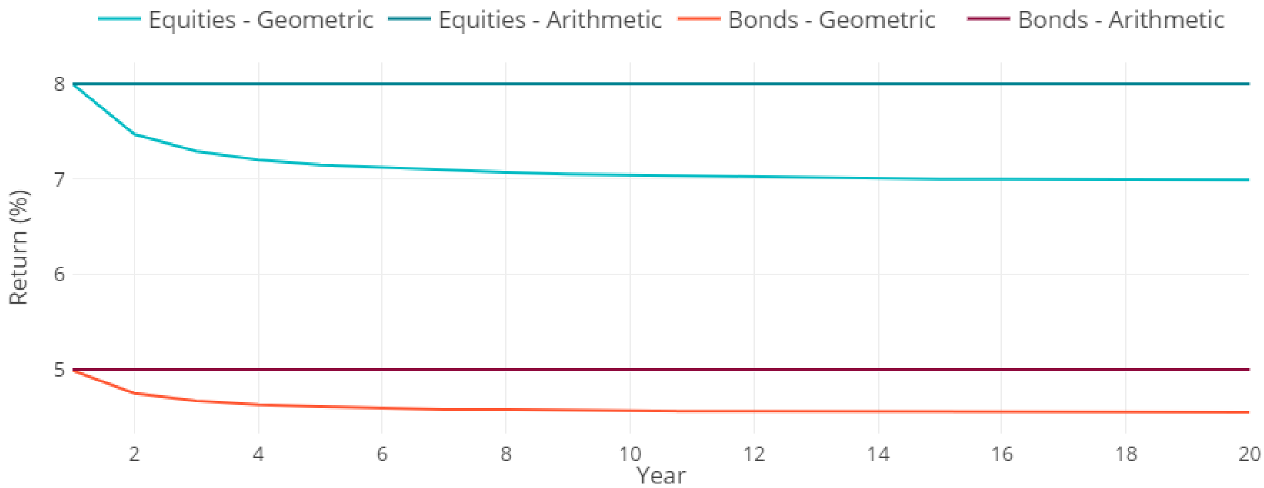
As an example of how time and volatility impacts variance drain, consider the following simple two asset example (Equities and Bonds). Asset class assumptions are given in Table 1.

Table 1: Asset class assumptions

	Equities	Bonds
Expected arithmetic return	8.0%	5.0%
Standard deviation	15.0%	10.0%
Cross-asset correlation	0.0%	

We run 1,000,000 simulations with these assumptions and plot the arithmetic and geometric returns of our assets in Figure 1. Clearly, the variance drain increases over time, and is more significant for the more volatile Equities asset class. After 3 years the expected return for Equities is 7.3%, and for Bonds is 4.7%. Expected returns drop to 7.1% and 4.6% respectively over 5 years, and to 7.0% and 4.5% over 20 years.

Figure 1: Variance drain



Source: Jacobi. Simulated results.

These results are well-known amongst professional investors, but sometimes catch new entrants by surprise when using new products such as leveraged ETFs. What is more interesting, and not as well understood, is the relationship between the geometric returns of individual asset classes, and a portfolio of those asset classes, over multi-period horizons.

2. Expected returns in multi-period, multi-asset models

Many investors performing simulation-based, multi-period analysis intuitively expect the long-term returns of a multi-asset portfolio to equal the expected geometric return of the assets multiplied by the asset weights. While this expectation is true for a one period model, it is not necessarily true when compounding returns over multiple periods. In practice, the volatility of the assets, correlation of the assets and the approach to rebalancing place an upward bias on portfolio returns over a multi-period horizon.

To illustrate this outcome, consider a simple two-asset portfolio consisting of Equities and Bonds, each with a 50% allocation at the outset. Equities and Bonds are assumed to have the return and volatility assumptions shown in Table 1.

An intuitive 5-year portfolio return expectation under these assumptions is 5.9%, calculated as:

$$5.9\% = 0.5 \times 7.1\% + 0.5 \times 4.6\%$$

When investors calculate portfolio returns using this approach, they are typically assuming the portfolio is rebalanced to target weight at the end of each period. An alternative approach that we also consider is a portfolio that is not rebalanced over time. This gives us three ways to calculate portfolio return:

1. Sum product of the portfolio target weights and asset geometric returns
2. Geometric return of the rebalanced portfolio derived from Monte Carlo analysis
3. Geometric return of the non-rebalanced portfolio derived from Monte Carlo analysis

N.B. For more technical information on how these returns are calculated please refer to the Appendix.

This paper investigates the relationship between these three approaches via a simulation study assuming a five-year investment horizon, made up of 5 investment periods, each of length one year. We use the return, volatility, correlation, and asset allocation assumptions identified in Table 1.

Results from 1,000,000 simulations using these assumptions are shown in Table 2 below.

Table 1: Simulation results

Approach	Method	Average Geometric Return
1	Sum product of weights and returns	0.0589
2	Rebalanced	0.0620
3	Not rebalanced	0.0622

The portfolio weighted geometric return is noticeably lower than either the rebalanced or the non-rebalanced result. The non-rebalanced return is marginally higher than the rebalanced return due to the positive skewness in the cumulative return distribution.

There is a considerable amount of research written on the historical impact of holding a rebalanced portfolio (Approach 2) versus a non-rebalanced portfolio (Approach 3), and on determining the best approach to rebalancing in an expectations sense. Very little has been written on the factors that contribute to the difference between the return estimated by approaches (1) and (2). The remainder of this paper investigates those factors.

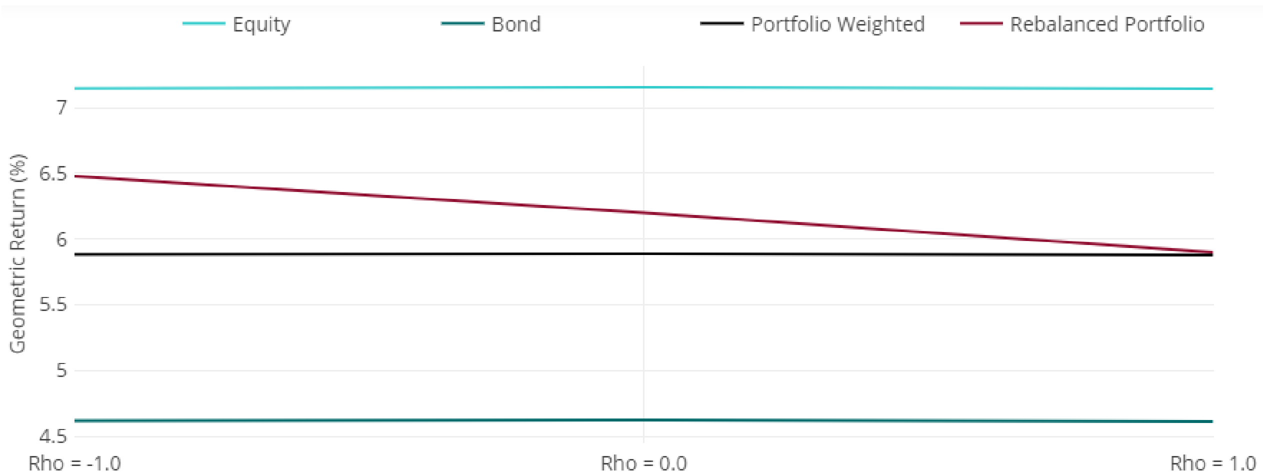
3. Causal factors

The three main contributors to this effect, assuming fixed asset allocation, are the degree of correlations amongst the assets, the volatility of the assets, and whether returns are assumed to be mean reverting. Changing the asset allocation will also influence the effect by changing the emphasis on the other causal factors.

a) Correlation

The higher the correlation between the two assets, the closer it becomes to a single asset portfolio, and the difference between approaches (1) and (2) tends to zero. The black line in the following chart corresponds to the portfolio weighted geometric average. This effect can be seen in the converging relationship between the red (portfolio returns derived from simulations) and black lines.

Figure 2: Changing correlation between assets

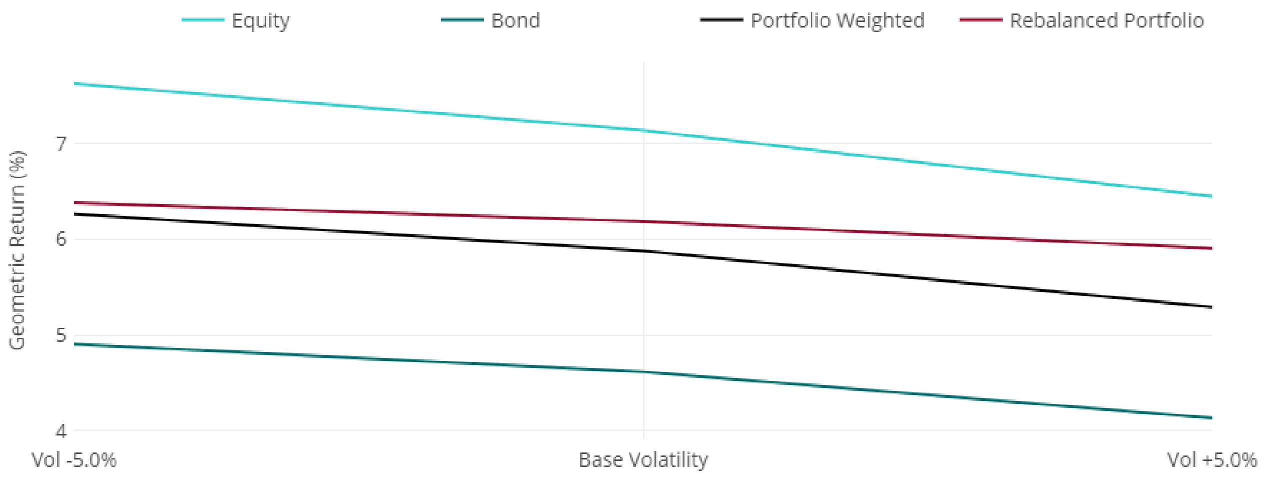


Source: Jacobi. Simulated results.

b) Volatility

Figure 3 shows two additional scenarios: a high volatility scenario where the asset volatilities are increased by +5% relative to the base assumptions, and a low volatility scenario where the asset volatilities are decreased by -5%. The black line again shows the portfolio weighted geometric average. Note, as we change the asset volatility, the geometric return changes (higher volatility causes a lower geometric return). As asset volatility increases, the red and black lines diverge meaning the difference between the rebalanced portfolio return and the portfolio weighted geometric return gets larger.

Figure 3: Changing volatility of assets

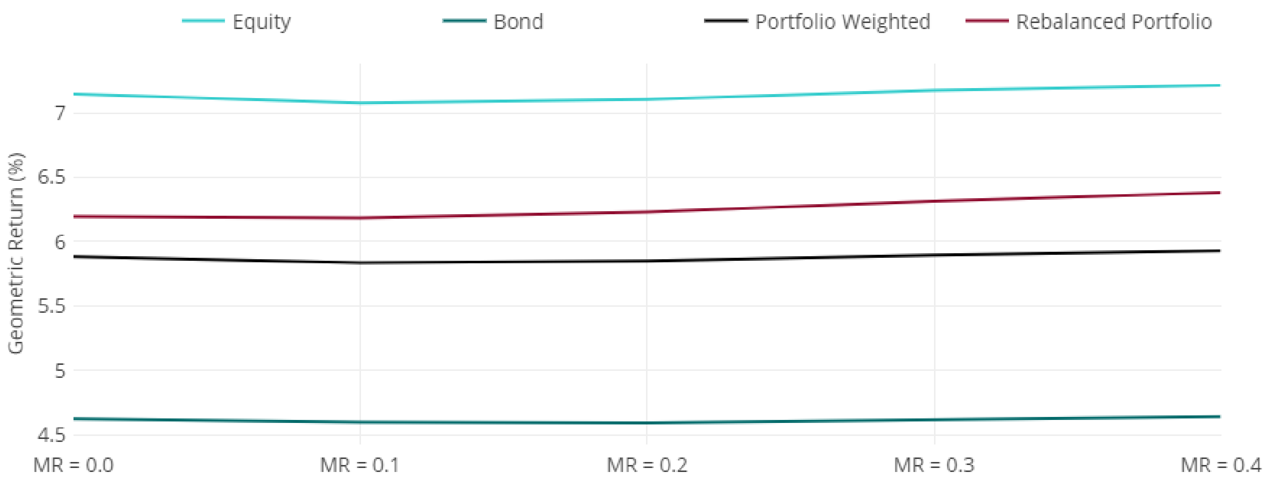


Source: Jacobi. Simulated results.

c) Mean Reversion

Introducing a basic mean reversion (MR) model to trend growth is explored in this section. In this model, any above or below trend growth is partially corrected in the following period by the amount of assumed MR. Note that an MR value of 0.5 means that 50% of the cumulative over or undervaluation at the end of one period will be corrected in the following period (this process is independent of idiosyncratic noise for the following period). As MR increases, the difference between the rebalanced return and the weighted geometric return increases (again shown through deviation between black and red line in Figure 4). Note also that over long horizons, mean reversion reduces variability in the terminal wealth distribution, thereby resulting in higher geometric returns, and explains why the series have a positive slope.

Figure 4: Changing mean reversion of returns



Source: Jacobi. Simulated results.

4. Practicalities for forecasting

This analysis shows that the intuitive approach to forecasting portfolio returns over multiple periods – the sum product of their asset weights and returns – is not an investable outcome. However, there are also practicalities that limit an investor's ability to earn the returns calculated under a simulation approach. For instance, there are costs associated with rebalancing such as buy/sell spreads and tax realisation that need to be accounted for. It is also impractical to assume that some asset classes can be precisely rebalanced each period (private equity for instance). If a portfolio has a high allocation to volatile, illiquid asset classes like private equity the assumption that asset weights can freely be rebalanced each year will potentially lead to a large 'rebalance premium' and overstated expected returns. So what are investors to do?

At Jacobi, we believe that the answer lies in having the tools to fully customise asset class and portfolio return models. The old one period, one-size-fits-all approach to portfolio modelling is not suited to addressing the challenges multi-asset investors face today. Investors need to be able to test and tailor their assumptions about return distributions and portfolio rebalancing to their own approach, and be able to incorporate illiquid assets in an intelligent way. This could mean adding variable rebalance costs for different assets, or incorporating a lag into how particular assets are rebalanced to target for instance.

Conclusion

- Issues like sequencing risk, maturing defined benefit plans, and the rise of defined contribution have seen investors pay more attention to the implications of time series return modelling. This shift brings into a focus a range of interactions between assumptions that affect portfolio return and volatility.

- One often unexpected result of multi-period, simulation-based return modelling is that the expected return from a portfolio of assets (whether rebalanced or not) is higher than the result obtained by multiplying and summing the average geometric return of the assets by their portfolio weights. The factors influencing this outcome are asset correlations, asset volatility and mean reversion.

- The return calculated by multiplying and summing the average geometric return of the assets by their portfolio weights is not an investable outcome. However, there are also practicalities that limit an investor's ability to earn the returns calculated under a simulation approach.

- When modelling asset class and portfolio returns over time investors need to be able to set intelligent assumptions about return distributions, rebalancing and mean reversion, and be able to handle illiquid assets in an intelligent way. The old one period, one-size-fits-all approach to portfolio modelling is not suited to addressing the challenges multi-asset investors face today.

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For more information on Jacobi's modelling framework or other tools available within our portfolio modelling and visualization suite, please do not hesitate to contact us.

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Appendix

In this paper we discussed three approaches to calculating the multi-period portfolio return:

1. Sum product of the portfolio target weights and asset geometric returns
2. Geometric return of the rebalanced portfolio derived from Monte Carlo analysis
3. Geometric return of the non-rebalanced portfolio derived from Monte Carlo analysis

Below we show the mathematical representation of these returns assuming a two-period investment horizon to keep the equations more readable (although we can easily extend this to a higher number of periods). We assume a two-asset portfolio consisting of Equities and Bonds, each with a 50% allocation.

Equation (1) shows the sum product of the portfolio target weights and the individual asset geometric returns. That is, a portfolio weighted geometric average of the underlying assets.

$$r_{port\ weigt\ geo} = 0.5\left[\sqrt[2]{(1+r_{e,1})(1+r_{e,2})}-1\right] + 0.5\left[\sqrt[2]{(1+r_{b,1})(1+r_{b,2})}-1\right] \quad (1)$$

where r_e and r_b refer to the returns on equities and bonds respectively, and the numbers indicate to which period the returns relate.

While many investors use this approach to estimate the return of their portfolios over multiple periods, it is not an investable outcome.

Equation (2) shows the portfolio geometric return, when the portfolio is rebalanced after each period

$$r_{geo}^{(rebalanced)} = \sqrt[2]{(1+0.5r_{e,1}+0.5r_{b,1}) \times (1+0.5r_{e,2}+0.5r_{b,2})} - 1 \quad (2)$$

Equation (3) shows the portfolio geometric return, when the portfolio is not rebalanced, but has a 50% allocation to each asset at inception.

$$r_{geo}^{(non-rebalanced)} = \sqrt[2]{0.5(1+r_{e,1})(1+r_{e,2})+0.5(1+r_{b,1})(1+r_{b,2})} - 1$$
(3)